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Exact Vibration Solutions for Nonuniform Timoshenko Beams with Attachments

Sen Yung Lee*
National Cheng Kung University, Tainan, Taiwan 701, Republic of China
and
Shueei Muh Lin†
Kung Shan Institute of Technology, Tainan, Taiwan 710, Republic of China

The exact solution for the free vibration of a symmetric nonuniform Timoshenko beam with tip mass at one end and elastically restrained at the other end of the beam is derived. The two coupled governing characteristic differential equations are reduced into one complete fourth-order ordinary differential equation with variable coefficients in the angle of rotation due to bending. The frequency equation is derived in terms of the four normalized fundamental solutions of the differential equation. It can be shown that, if the coefficients of the reduced differential equation can be expressed in polynomial form, the exact fundamental solutions can be found by the method of Frobenius. Finally, several limiting cases are studied and the results are compared with those in the existing literature.

Nomenclature

$A(x)$ = cross-sectional area of the beam
$E(x)$ = Young's modulus of beam material
$G(x)$ = shear modulus of beam material
$I(x)$ = area moment inertia of the beam
$J(x)$ = mass moment of inertia of the beam per unit length
J_m = rotatory inertia attached at the right end of the beam
K_T, K_θ = translational and rotational spring constants at the left end of the beam, respectively
L = length of the beam
M = concentrated mass attached at the right end of the beam
M_b = total mass of the beam
$m(x)$ = mass of the beam per unit length
$Q(x)$ = beam shear rigidity, $\kappa G(x)A(x)$
$q(\xi)$ = dimensionless shear rigidity, $Q(\xi)/Q(0)$
$R(x)$ = beam bending rigidity, $E(x)I(x)$
$r(\xi)$ = dimensionless bending rigidity, $R(\xi)/R(0)$
$s(\xi)$ = dimensionless mass, $m(\xi)/m(0)$
$v(\xi)$ = dimensionless mass moment inertia, $J(\xi)/J(0)$
x = length variable of the beam
Y = beam lateral displacement
y = dimensionless lateral displacement, Y/L
α = dimensionless rotatory inertia of the attached mass, $J_m/[m(0)L^3]$
β_T, β_θ = dimensionless translational and rotational spring constants, respectively, $K_TL^3/R(0)$, $K_\theta L/R(0)$
γ = dimensionless concentrated mass, $M/[m(0)L]$
δ = dimensionless ratio of bending rigidity to shear rigidity at $x = 0$, $R(0)/[Q(0)L^2]$
η = dimensionless ratio of mass moment inertia to mass at $x = 0$, $J(0)/[m(0)L^2]$
κ = shear correction factor of the beam
λ = taper ratio of the beam
μ = dimensionless ratio of attached mass to total mass of the beam, M/M_b
ξ = dimensionless distance to the left end of the beam, x/L
ω = angular frequency of beam vibration
Ψ = angle of rotation due to bending
Ω^2 = dimensionless frequency, $m(0)\omega^2L^4/R(0)$

Introduction

NONUNIFORM beams are widely used in many structural applications in order to optimize the distribution of weight and strength and sometimes to satisfy special architectural and functional requirements. Therefore, the analysis of nonuniform beams is of interest to many mechanical, aeronautical, and civil engineers.

It is a standard engineering practice to analyze beams of uniform or variable properties on the basis of Bernoulli-Euler beam theory. However, if the effect of shear distortion and rotatory inertia is considered, then a higher-order beam theory (Timoshenko beam theory) is required. Based on Bernoulli-Euler beam theory, the analysis of nonuniform beams has been studied by many authors via many different methods. A brief review of the work can be found in the work recently done by Lee and Kuo.[1,2] They made the static and dynamic analysis of a general elastically restrained nonuniform Bernoulli-Euler beam. The exact solution for the problem governed by a general self-adjoint fourth-order ordinary differential equation with arbitrarily polynomial varying coefficients were derived in Green's function form and concisely expressed in terms of the four normalized fundamental solutions of the system. Exact stiffness matrices for the analysis of nonuniform Bernoulli-Euler beams were developed by Karabalis and Beskos.[3]

For Timoshenko beams the governing characteristic differential equations are two coupled differential equations expressed in terms of two dependent variables: the flexural displacement and the angle of rotation due to bending. It is well known that, if a beam is uniform, then the two coupled differential equations can be uncoupled into two complete fourth-order ordinary differential equations in the flexural displacement and the angle of rotation due to bending.[4,5] However, this is not the case for nonuniform beams. Consequently, exact solutions for the problems were never given, and the problems were mainly studied by approximate methods such as the finite element method,[6] the optimized Rayleigh-Ritz method,[7] and the transfer matrix method.[8]

In this paper the exact solution for the free vibration of a symmetric nonuniform Timoshenko beam with tip mass at

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*Professor, Mechanical Engineering Department.
†Lecturer, Mechanical Engineering Department.

one end and elastically restrained at the other end of the beam is derived. The two coupled governing characteristic differential equations are reduced into one complete fourth-order ordinary differential equation with variable coefficients in the angle of rotation due to bending. The frequency equation is derived in terms of the four normalized fundamental solutions of the characteristic differential equation. It can be shown that, if the coefficients of the reduced differential equation are in polynomial form, then the exact fundamental solutions can be obtained by the method of Frobenius. The limiting cases such as uniform Timoshenko beams, nonuniform Rayleigh beams, and Bernoulli-Euler beams are also examined. Finally, several examples are given to illustrate the validity and accuracy of the analysis.

Analysis

Nonuniform Timoshenko Beams

Consider the free vibration of a symmetric nonuniform Timoshenko beam with tip mass at one end and elastically restrained at the other end of the beam (see Fig. 1). For time harmonic vibration with angular frequency ω, the dimensionless governing characteristic differential equations of motion are

$$\frac{d}{d\xi}\left[\frac{q}{\delta}\left(\frac{dy}{d\xi}-\Psi\right)\right]+s\Omega^2 y=0 \tag{1}$$

$$\frac{d}{d\xi}\left(r\frac{d\Psi}{d\xi}\right)+\frac{q}{\delta}\left(\frac{dy}{d\xi}-\Psi\right)+\nu\eta\Omega^2\Psi=0, \qquad \xi\in(0,1) \tag{2}$$

and the following associated boundary conditions:

At $\xi = 0$:

$$\beta_T y=\frac{1}{\delta}\left(\frac{dy}{d\xi}-\Psi\right) \tag{3}$$

$$\beta_\theta\Psi=\frac{d\Psi}{d\xi} \tag{4}$$

At $\xi = 1$:

$$\alpha\Omega^2\Psi=r\frac{d\Psi}{d\xi} \tag{5}$$

$$\gamma\Omega^2 y=\frac{q}{\delta}\left(\frac{dy}{d\xi}-\Psi\right) \tag{6}$$

Differentiating Eq. (2) and then combining it with Eq. (1), one obtains the following relation between the dimensionless flexural displacement and the angle of rotation due to bending:

$$y=\frac{1}{s\Omega^2}\left[\frac{d^2}{d\xi^2}\left(r\frac{d\Psi}{d\xi}\right)+\frac{d}{d\xi}(\nu\eta\Omega^2\Psi)\right] \tag{7}$$

Substituting the aforementioned relation back into Eq. (2), one obtains the governing characteristic differential equation of motion, which is a fourth-order ordinary differential equation with variable coefficients, in terms of the angle of rotation due to bending:

$$q\frac{d}{d\xi}\left\{\frac{1}{s\Omega^2}\left[\frac{d^2}{d\xi^2}\left(r\frac{d\Psi}{d\xi}\right)+\frac{d}{d\xi}(\nu\eta\Omega^2\Psi)\right]\right\}$$
$$+\delta\frac{d}{d\xi}\left(r\frac{d\Psi}{d\xi}\right)+(\delta\nu\eta\Omega^2-q)\Psi=0, \qquad \xi\in(0,1) \tag{8}$$

The associated dimensionless boundary conditions become at $\xi = 0$:

$$\frac{\beta_T}{\Omega^2}\left[\frac{d^2}{d\xi^2}\left(r\frac{d\Psi}{d\xi}\right)+\frac{d}{d\xi}(\nu\eta\Omega^2\Psi)\right]=-\left[\frac{d}{d\xi}\left(r\frac{d\Psi}{d\xi}\right)+\nu\eta\Omega^2\Psi\right] \tag{9}$$

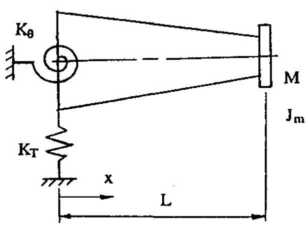

Fig. 1 Nonuniform beam system.

$$\beta_\theta\Psi=\frac{d\Psi}{d\xi} \tag{10}$$

At $\xi = 1$:

$$\alpha\Omega^2\Psi=r\frac{d\Psi}{d\xi} \tag{11}$$

$$\frac{\gamma}{s}\left[\frac{d^2}{d\xi^2}\left(r\frac{d\Psi}{d\xi}\right)+\frac{d}{d\xi}(\nu\eta\Omega^2\Psi)\right]=-\left[\frac{d}{d\xi}\left(r\frac{d\Psi}{d\xi}\right)+\nu\eta\Omega^2\Psi\right] \tag{12}$$

Uniform Timoshenko Beams

For uniform Timoshenko beams, $q = r = s = \nu = 1$. The governing characteristic differential equation (8) is reduced to

$$\frac{d^4\Psi}{d\xi^4}+\Omega^2(\eta+\delta)\frac{d^2\Psi}{d\xi^2}+(-\Omega^2+\eta\delta\Omega^4)\Psi=0 \tag{13}$$

By differentiating Eqs. (1) and (7) with respect to dimensionless spatial variable ξ twice, one has

$$\frac{d^3\Psi}{d\xi^3}=\frac{d^4 y}{d\xi^4}+\delta\Omega^2\frac{d^2 y}{d\xi^2} \tag{14}$$

$$\frac{d^2 y}{d\xi^2}=\frac{1}{\Omega^2}\left(\frac{d^5\Psi}{d\xi^5}+\eta\Omega^2\frac{d^3\Psi}{d\xi^3}\right) \tag{15}$$

Upon differentiating Eq. (13) once and substituting Eqs. (1), (14) and (15) into it, one can express the governing characteristic differential equation in terms of the dimensionless flexural displacement, which is exactly the same as the one given by Huang[4]:

$$\frac{d^4 y}{d\xi^4}+\Omega^2(\eta+\delta)\frac{d^2 y}{d\xi^2}+(-\Omega^2+\eta\delta\Omega^4)y=0 \tag{16}$$

Nonuniform Rayleigh Beams

For Rayleigh beams the effect of rotatory inertia is considered and that of shear deformation is neglected. By letting $G\to\infty$ and $\delta = 0$ in Eq. (8) and dividing them by a factor q, in terms of the angle of rotation due to bending, the governing characteristic differential equation is obtained:

$$\frac{d}{d\xi}\left\{\frac{1}{s\Omega^2}\left[\frac{d^2}{d\xi^2}\left(r\frac{d\Psi}{d\xi}\right)+\frac{d}{d\xi}(\nu\eta\Omega^2\Psi)\right]\right\}-\Psi=0 \tag{17}$$

The governing characteristic differential equation can also be expressed in terms of the dimensionless flexural displacement and derived from Eq. (17).

Expanding Eq. (17), one has

$$\left[\frac{d}{d\xi}\left(\frac{1}{s\Omega^2}\right)\right]\left[\frac{d^2}{d\xi^2}\left(r\frac{d\Psi}{d\xi}\right)+\frac{d}{d\xi}(\nu\eta\Omega^2\Psi)\right]$$
$$+\left(\frac{1}{s\Omega^2}\right)\frac{d}{d\xi}\left[\frac{d^2}{d\xi^2}\left(r\frac{d\Psi}{d\xi}\right)+\frac{d}{d\xi}(\nu\eta\Omega^2\Psi)\right]-\Psi=0 \tag{18}$$

Upon substituting Eq. (7) into the second term, letting $\Psi = dy/d\xi$ for the other terms in Eq. (18), and then dividing by $d[1/(s\Omega^2)]/d\xi$, one obtains

$$\frac{d^2}{d\xi^2}\left(r\frac{d^2y}{d\xi^2}\right) + \frac{d}{d\xi}\left[(\nu\eta\Omega^2)\frac{dy}{d\xi}\right] - s\Omega^2 y = 0 \tag{19}$$

It should be mentioned here that Eq. (19) can also be obtained directly from Eq. (7) by letting $\Psi = dy/d\xi$.

Nonuniform Bernoulli-Euler Beams

For Bernoulli-Euler beams both shear deformation and rotatory inertia are neglected. By letting $\nu\eta = 0$ in Eqs. (17) and (19), the governing differential equation becomes

$$\frac{d}{d\xi}\left[\frac{1}{s\Omega^2}\frac{d^2}{d\xi^2}\left(r\frac{d\Psi}{d\xi}\right)\right] - \Psi = 0 \tag{20}$$

or

$$\frac{d^2}{d\xi^2}\left(r\frac{d^2y}{d\xi^2}\right) - s\Omega^2 y = 0 \tag{21}$$

The associated boundary conditions for the uniform Timoshenko beam and nonuniform Rayleigh and Bernoulli-Euler beams can be obtained from Eqs. (3-6) or (9-12) by performing suitable limit procedures.

Frequency Equation

If the four linearly independent fundamental solutions $V_j(\xi)$, $j = 1, 2, 3, 4$, of the corresponding governing characteristic differential equation are chosen such that they satisfy the following normalization conditions at the origin of the coordinate system:

$$\begin{bmatrix} V_1(0) & V_2(0) & V_3(0) & V_4(0) \\ V_1'(0) & V_2'(0) & V_3'(0) & V_4'(0) \\ V_1''(0) & V_2''(0) & V_3''(0) & V_4''(0) \\ V_1'''(0) & V_2'''(0) & V_3'''(0) & V_4'''(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$

where primes indicate differentiation with respect to the dimensionless spatial variable ξ, then after substituting the homogeneous solution (which is a linear combination of the fundamental solution of the characteristic differential equation into the associated boundary conditions), one obtains the frequency equation of the system:

$$\pi = 0 \tag{23}$$

The frequency equations in the angle of rotation due to bending for the Timoshenko beam with general elastically restraints at one end and three limiting cases of the general system are tabulated in Appendix A.

The frequency equations in the dimensionless flexural displacement for the Rayleigh beam with general elastic restraints at one end and three limiting cases are tabulated in Appendix B.

Normalized Fundamental Solutions

In the previous sections the governing characteristic differential equations of motion for the Timoshenko, Rayleigh, and Bernoulli-Euler beams were derived in the form of a fourth-order differential equation with variable coefficients. The corresponding frequency equation is expressed in terms of the four normalized fundamental solutions of the associated differential equation. However, in general, the closed-form fundamental solutions of a fourth-order differential equation with variable coefficients are not available. However, if the coefficients of the equation, which involve the material properties and/or geometric parameters, can be expressed in polynomial form, then a power series representation of the fundamental solutions can be constructed by the method of Frobenius.

Upon expanding the governing characteristic differential equations of motion for the Timoshenko, Rayleigh, and Bernoulli-Euler beams, they will take the following form:

$$p_4(\xi)\frac{d^4V(\xi)}{d\xi^4} + p_3(\xi)\frac{d^3V(\xi)}{d\xi^3} + p_2(\xi)\frac{d^2V(\xi)}{d\xi^2} + p_1(\xi)\frac{dV(\xi)}{d\xi} + p_0(\xi)V(\xi) = 0 \qquad \xi \in (0, 1) \tag{24}$$

where V may represent either the angle of rotation due to bending or the dimensionless flexural displacement. If the coefficients of the differential equation (24) are given in the following polynomial form,

$$p_4 = \sum_{j=0}^{n_4} a_j\xi^j, \quad p_3 = \sum_{j=0}^{n_3} b_j\xi^j, \quad p_2 = \sum_{j=0}^{n_2} c_j\xi^j$$
$$p_1 = \sum_{j=0}^{n_1} d_j\xi^j, \quad p_0 = \sum_{j=0}^{n_0} e_j\xi^j, \tag{25}$$

where n_f, $f = 0, 1, 2, 3, 4$, are integers representing the number of terms in the series, then one can assume that the four fundamental solutions of Eq. (24) are in the form of

$$V_i = \sum_{j=0}^{\infty} k_{i,j}\,\xi^j, \qquad i = 1, 2, 3, 4 \tag{26}$$

and

$$\begin{aligned} &\text{For} \quad V_1(\xi){:}\ k_{1,0} = 1, && k_{1,1} = k_{1,2} = k_{1,3} = 0 \\ &\text{For} \quad V_2(\xi){:}\ k_{2,1} = 1, && k_{2,0} = k_{2,2} = k_{2,3} = 0 \\ &\text{For} \quad V_3(\xi){:}\ k_{3,2} = 1/2, && k_{3,0} = k_{3,1} = k_{3,3} = 0 \\ &\text{For} \quad V_4(\xi){:}\ k_{4,3} = 1/6, && k_{4,0} = k_{4,1} = k_{4,2} = 0 \end{aligned} \tag{27}$$

These four fundamental solutions satisfy the normalization condition (22). Upon substituting Eq. (26) into Eq. (24) and collecting the coefficients of like powers of ξ, the following recurrence formula can be obtained:

$$\begin{aligned} k_{i,z+4} = \frac{-1}{(z+4)(z+3)(z+2)(z+1)a_0}\Bigg[&\sum_{j=0}^{z} e_j k_{i,z-j} \\ &+ \sum_{j=0}^{z}(z-j+1)d_j k_{i,z-j+1} + \sum_{j=0}^{z}(z-j+2)(z-j+1)c_j k_{i,z-j+2} \\ &+ \sum_{j=0}^{z}(z-j+3)(z-j+2)(z-j+1)b_j k_{i,z-j+3} \\ &+ \sum_{j=1}^{z}(z-j+4)(z-j+3)(z-j+2)(z-j+1)a_j k_{i,z-j+4}\Bigg] \\ &i = 1, 2, 3, 4, \qquad z = 0, 1, 2, 3, \ldots \end{aligned} \tag{28}$$

With this recurrence formula, one can generate the four exact normalized fundamental solutions of the differential equation (24). Consequently, upon substituting these fundamental solutions into the associated frequency equation, the natural frequencies of the beams are obtained.

Verification and Examples

The following examples are given to illustrate the validity and accuracy of the analysis.

Example 1: Table 1 shows the first four dimensionless natural frequencies of a Timoshenko beam with constant width and linearly varying thickness, clamped at one end and carrying a concentrated mass at the other end. The material properties of the beam are assumed to be constants. Consequently, $q = s = (1 + \lambda\xi)$ and $r = \nu = (1 + \lambda\xi)^3$, where λ is the taper ratio of the beam. Here, the rotatory inertia of the tip mass is neglected and $\mu = M/M_b$, where M_b is the total mass of the beam. The results obtained by the present analysis are compared with those given by Rossi et al.,[6] who evaluated the frequencies by the finite element method. It can be observed that the difference between both the convergent solutions and those given by Rossi et al.[6] are less than 0.6%.

Example 2: Table 2 shows the first five dimensionless natural frequencies of nonuniform beams with tip mass at the right end are shown. Both the width and the depth of the beams vary linearly with the same taper ratio λ. The material properties of the beam are assumed to be constants. Consequently, $q = s = (1 + \lambda\xi)^2$ and $r = v = (1 + \lambda\xi)^4$. The natural frequencies are evaluated via Timoshenko, Rayleigh, and Bernoulli-Euler beam theories. It can be observed that the natural frequencies of the Rayleigh and the Bernoulli-Euler beams determined by employing the frequency equation expressed in terms of the angle of rotation due to bending, given in Appendix A, are the same as those determined by employing the frequency equation expressed in terms of the flexural displacement, given in Appendix B. A comparison of the natural frequencies of the Bernoulli-Euler beams with those given by Lau[9] shows that the results are very consistent.

Example 3: For clamped-free uniform Timoshenko beams, the governing differential equation is Eq. (13). The associated four normalized fundamental solutions are exact and are given as

$$V_1 = \frac{1}{\epsilon^2 + \zeta^2}(\zeta^2 \cosh \epsilon\xi + \epsilon^2 \cos \zeta\xi)$$
$$V_2 = \frac{1}{\epsilon^2 + \zeta^2}\left(\frac{\zeta^2}{\epsilon} \sinh \epsilon\xi + \frac{\epsilon^2}{\zeta} \sin \zeta\xi\right)$$
$$V_3 = \frac{1}{\epsilon^2 + \zeta^2}(\cosh \epsilon\xi - \cos \zeta\xi)$$
$$V_4 = \frac{1}{\epsilon^2 + \zeta^2}\left(\frac{1}{\epsilon} \sinh \epsilon\xi - \frac{1}{\zeta} \sin \zeta\xi\right) \tag{29}$$

where

$$\epsilon = \sqrt{\frac{-A + \sqrt{A^2 - 4B}}{2}}$$
$$\zeta = \sqrt{\frac{A + \sqrt{A^2 - 4B}}{2}} \tag{30}$$

where $A = \Omega^2(\eta + \delta)$ and $B = -\Omega^2 + \eta\delta\Omega^4$.

Upon substituting the four fundamental solutions (29) into the corresponding frequency equation listed in Appendix A, one obtains

$$2 + [\Omega^2(\eta - \delta)^2 + 2]\cosh \epsilon \cos \zeta - \frac{\Omega(\eta + \delta)}{(1 - \eta\delta\Omega^2)^{1/2}} \sinh \epsilon \sin \zeta = 0 \tag{31}$$

The frequency equation is exactly the same as the one given by Huang.[4]

Conclusion

In this paper the exact solution for the free vibrations of a symmetric nonuniform Timoshenko beam with tip mass at one end and elastically restrained at the other end of the beam are derived. The two coupled governing characteristic differential equations are reduced into one complete fourth-order ordinary differential equation with variable coefficients in the angle of rotation due to bending. The frequency equation is derived in terms of the four normalized fundamental solutions of the differential equation. It is shown that, if the coefficients of the reduced differential equation are in polynomial form, then the exact fundamental solutions can be obtained.

In the present analysis only the free vibration of the system is studied. It is of interest to extend the present study to the problems of forced vibrations.

Table 1 First four dimensionless frequencies of a cantilever nonuniform Timoshenko beam with attached mass at the right end of the beam [$q = s = (1 - 0.2\,\xi)$, $r = v = (1 - 0.2\,\xi)^3$]

			N[a]						
η	μ	Ω	15	20	25	30	35	40	#[b]
0.0016	0.2	Ω_1	2.59	2.59					2.59
		Ω_2	15.68	15.67	15.67				15.67
		Ω_3	34.97	41.30	41.53	41.53			41.56
		Ω_4	63.76	62.48	72.84	75.63	75.66	75.66	75.84
	0.6	Ω_1	1.85	1.85					1.85
		Ω_2	14.44	14.44					14.43
		Ω_3	34.08	39.87	40.07	40.07			40.10
		Ω_4	63.57	62.07	71.83	74.22	74.24	74.24	74.37
0.01	0.2	Ω_1	2.46	2.46					2.46
		Ω_2	12.27	12.27					12.27
		Ω_3	26.16	27.77	27.73	27.73			27.78
		Ω_4	29.98	36.54	44.06	44.89	44.89		45.15
	0.6	Ω_1	1.77	1.77					1.77
		Ω_2	11.42	11.42					11.42
		Ω_3	25.10	26.81	26.79	26.79			26.84
		Ω_4	30.20	36.25	43.37	44.12	44.12		44.39

[a]Number of terms of power series taken.
[b]By Rossi et al.[6]

Table 2 First five dimensionless frequencies of a nonuniform cantilever beam with attached mass at the right end of the beam [$q = s = (1 - 0.1\,\xi)^2$, $r = v = (1 - 0.1\,\xi)^4$]

		Timoshenko	Rayleigh		Bernoulli-Euler		
$\alpha = \gamma$		$\eta = 0.0008$, $\delta = 0.0025$	$\eta = 0.0008$, $\delta = 0$		$\eta = 0$, $\delta = 0$		
		a	a	b	a	b	c
0.0	$\Omega_1^{1/2}$	1.9095	1.9150	1.9150	1.9167	1.9167	1.9167
	$\Omega_2^{1/2}$	4.5359	4.6162	4.6162	4.6422	4.6422	4.6422
	$\Omega_3^{1/2}$	7.3093	7.5917	7.5917	7.6934	7.6934	7.6934
	$\Omega_4^{1/2}$	9.8552	10.486	10.486	10.742	10.742	10.742
	$\Omega_5^{1/2}$	12.187	13.288	13.288	13.797	13.797	13.797
0.2	$\Omega_1^{1/2}$	1.2906	1.2910	1.2910	1.2912	1.2912	1.2912
	$\Omega_2^{1/2}$	2.2670	2.2813	2.2813	2.2820	2.2820	2.2820
	$\Omega_3^{1/2}$	4.9001	5.0216	5.0216	5.0375	5.0375	5.0375
	$\Omega_4^{1/2}$	7.5166	7.9011	7.9011	7.9745	7.9745	7.9745
	$\Omega_5^{1/2}$	9.9660	10.770	10.770	10.969	10.969	10.969

[a]Determined from the frequency equation in the angle of rotation due to bending.
[b]Determined from the frequency equation in the dimensionless flexural displacement.
[b]By Lau.[9]

Appendix A

Table A1 displays the frequency equation of nonuniform Timoshenko beams in the angle of rotation due to bending.

Table A1 Frequency equation of nonuniform Timoshenko beams

Case	K_T	K_θ	Frequency equation, $\pi = 0$
1	Const	Const	$\pi = -(g_1F_3H_1 + g_4F_4H_3 + g_2F_1H_4 - g_4F_3H_4 - g_2F_4H_1 - g_1F_1H_3) + g_5(g_1F_3H_2 + g_3F_4H_3 + g_2F_2H_4 - g_3F_3H_4 - g_2F_4H_2 - g_1F_2H_3)$ where $g_1 = \beta_T, \quad g_2 = 2r'(0)\beta_T + \Omega^2$ $g_3 = \beta_T[r''(0) + \eta\Omega^2] + r'(0)\Omega^2$ $g_4 = v'(0)\beta_T\eta\Omega^2 + \eta\Omega^4$ $g_5 = -\beta_\theta, \quad g_6 = -\alpha\Omega^2/r(1)$ $g_7 = r(1)\gamma, \quad g_8 = 2r'(1)\gamma + r(1)s(1)$ $g_9 = \gamma[r''(1) + v(1)\eta\Omega^2] + r'(1)s(1)$ $g_{10} = \eta\Omega^2[v'(1)\gamma + s(1)v(1)]$ $F_j = V_j'(1) + g_6V_j(1)$ $H_j = g_7V_j'''(1) + g_8V_j''(1) + g_9V_j'(1) + g_{10}V_j(1)$ $j = 1, 2, 3, 4$
2	∞	Const	π: same as that of Case 1 where $g_1 = 1, \quad g_2 = 2r'(0), \quad g_3 = r''(0) + \eta\Omega^2$ $g_4 = v'(0)\eta\Omega^2$ Other parameters: same as those of case 1
3	Const	∞	$\pi = g_1F_3H_2 + g_3F_4H_3 + g_2F_2H_4 - g_3F_3H_4 - g_2F_4H_2 - g_1F_2H_3$ All of the parameters: same as those of case 1
4	∞	∞	π: same as that of Case 3 where $g_1 = 1, \quad g_2 = 2r'(0), \quad g_3 = r''(0) + \eta\Omega^2$ $g_4 = v'(0)\eta\Omega^2$ Other parameters: same as those of case 1

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Appendix B

Table B1 displays the frequency equation of nonuniform Rayleigh beams in the dimensionless flexural displacement.

Table B1 Frequency equation of nonuniform Rayleigh beams

Case	K_T	K_θ	Frequency equation, $\pi = 0$
1	Const	Const	$\pi = -(F_2H_1 + g_3F_4H_2 + g_2F_1H_4 - g_3F_2H_4 - g_2F_4H_1 - F_1H_2) + g_4(F_3H_1 + g_3F_4H_3 + g_1F_1H_4 - g_3F_3H_4 - F_1H_3 - g_1F_4H_1)$ where $g_1 = r'(0), \quad g_2 = \eta\Omega^2, \quad g_3 = \beta_T$ $g_4 = -\beta_\theta$ $F_j = V_j''(1) - [\alpha\Omega^2/r(1)]V_j'(1)$ $H_j = r(1)V_j'''(1) + r'(1)V_j''(1) + v(1)\eta\Omega^2V_j'(1) + \gamma\Omega^2V_j(1), \quad j = 1, 2, 3, 4$
2	∞	Const	$\pi = -F_4H_2 + F_2H_4 + g_4F_4H_3 - g_4F_3H_4$ All of the parameters: same as those of case 1
3	Const	∞	$\pi = F_3H_1 + g_3F_4H_3 + g_1F_1H_4 - g_3F_3H_4 - F_1H_3 - g_1F_4H_1$ All of the parameters: same as those of case 1
4	∞	∞	$\pi = F_4H_3 - H_4F_3$ All of the parameters: same as those of case 1

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